Semiparametric Estimation of the Frequency of Unknown Periodic Functions and its Application to Laser Vibrometry Signals

Marc Lavielle and Céline Lévy-Leduc

Abstract—We propose a semiparametric approach to fundamental frequency estimation of an unknown periodic signal in additive white noise based on model selection. Our estimator maximizes a penalized version of the cumulated periodogram and is proved to be consistent and asymptotically efficient under very general conditions. When the number of observations is fixed, an implementation of this estimation method is proposed and illustrated on specific synthetic signals which arise in laser vibrometry. We extend this method for estimating the fundamental frequencies of two periodic functions having different fundamental frequencies when the data consist of their sum and additive white noise. We also compare the performances of our procedure with the so-called *microdoppler* technique, which is commonly used for laser vibrometry signals analysis. We show on simulated data that the penalized cumulated periodogram yields an accurate estimation of the frequencies at very low signal-to-noise ratios.

Index Terms—Frequency estimation, model selection, semiparametric estimation.

I. INTRODUCTION

ESTIMATING the fundamental frequency of a periodic signal in additive noise is a well-known problem that has already been addressed by several authors. Some key references in such a field are the book of Quinn and Hannan (see [16]) and a report written by Kootsookos (see [10]) in which many classical techniques concerning frequency estimation are explained. The different quoted authors focus on “multiharmonic frequency estimation” and “multitone frequency estimation” and propose, for most of them, a study of several performance indicators such as asymptotic efficiency, small sample size performance, computational complexity, and a study of the so-called *thresholding* phenomenon, which is a marked decrease in performance of the estimator for a relatively small change in signal-to-noise ratio (SNR) at a low SNR.

Most of the usual algorithms are devoted to signals with particular shapes, such as sinusoids or trigonometric polynomials. We can quote the estimator of Quinn and Fernandes described in [15], which is attractive since it requires fewer operations and

is more robust to initial conditions than maximum likelihood estimation. As for the one proposed by Lovell and Williamson in [13], it has the best performance among the weighted phase averaging estimators. In [9], Kay proposes an estimator which has similar performances for small noise levels but is biased in the presence of Gaussian noise. In [5], Carmona *et al.* propose practical methods for estimating the instantaneous frequency of some particular signals but more general than trigonometric polynomials.

Some theoretical results have also been obtained in parametric frameworks (i.e., when the number of unknown parameters is finite). In the case of a trigonometric polynomial, for example, the parameters are the coefficients and the fundamental frequency. Quinn and Thomson propose in [17] a consistent and asymptotically efficient estimator of the frequency of s in the following model:

$$x_j = s(j) + \varepsilon_j, \quad 1 \leq j \leq n \quad (1)$$

where s is a real trigonometric polynomial and the additive noise is not necessarily Gaussian. In [15], Quinn and Fernandes also propose a consistent and asymptotically efficient estimator of the frequency of s in the same model.

In this paper, we consider a sequence of observations (x_j) satisfying

$$x_j = s^*(j\Delta) + \varepsilon_{1,j} + i\varepsilon_{2,j}, \quad 1 \leq j \leq n \quad (2)$$

where s^* is an unknown complex-valued periodic function with unknown frequency f^* , Δ is the sampling period, and the $\varepsilon_{1,j}$'s, $\varepsilon_{2,j}$'s are independent Gaussian random variables of unknown variance σ^2 . Contrary to the previous mentioned approaches, we do not make any assumption on the shape of the unknown periodic signal s^* , this is, following the definitions given by Van der Vaart in [20], a semiparametric framework. Indeed, we aim to estimate a finite-dimensional parameter (here the frequency) when the other unknown parameters, called nuisance parameters, are infinite-dimensional (here, the function s^*). Then, all the parametric estimation methods no longer work for estimating f^* .

In such a semiparametric framework, some results have already been obtained by Golubev and Gassiat and Lévy-Leduc. In [8], a consistent and asymptotically efficient estimator of the period $1/f^*$ is proposed in the white noise model, which is the continuous version of model (2) when the observations x are real valued. In [7], Gassiat and Lévy-Leduc exhibit in model (2)

Manuscript received January 22, 2004; revised August 28, 2004. The work of the second author was supported by Thalès Optronique, Guyancourt, France. The associate editor coordinating the review of this manuscript and approving it for publication was Dr. Dominic K. C. Ho.

M. Lavielle is with the Université René Descartes, Paris, France, and also with the Université Paris-Sud, Orsay, France (e-mail: marc.lavielle@math.u-psud.fr).

C. Lévy-Leduc is with the Université Paris-Sud, Orsay, France. (e-mail: celine.levy-leduc@math.u-psud.fr).

Digital Object Identifier 10.1109/TSP.2005.849156

an estimator of the period $1/f^*$ having the same properties when the observations are real valued and $\Delta = \Delta_n$ goes to zero as n becomes large. These results are theoretical since the authors do not consider the practical implementation of their estimators.

Our aim is to propose an estimator that possesses both interesting theoretical and practical properties according to the performance indicators given by Kootsookos in [10]. Our method is based on the maximization of the *cumulated periodogram*, which is known to be asymptotically equivalent to the time-domain least-squares estimator (see [17]). The choice of the number of harmonics in the cumulated periodogram is obtained by introducing a penalization term. We prove that the estimator of f^* is consistent and asymptotically efficient. Furthermore, its practical implementation does not require heavy computations. Introduced by Akaike in [1], penalized criteria for model selection are widely used in many statistical frameworks such as the change-points detection problem (see [11]). Moreover, new theoretical results for model selection using penalized criteria have been recently obtained by Birgé and Massart in a nonasymptotic framework (see [3], [4]).

We illustrate our method on laser vibrometry signals. Laser vibrometry is a widely used technique for analyzing the vibrations of a vibrating object without any contact with it. For instance, it is used in medicine to study the heart ventricles vibrations, as well as in target identification (see [14]). We show that the algorithm accurately estimates the frequency of s^* in model (2), which can be seen as the vibration frequency of a target, at very low SNRs. This ensures that "long range" target identification is possible.

The method can be adapted to estimate the various frequencies in the following regression model:

$$x_j = \sum_{m=1}^M s_m^*(j\Delta) + \varepsilon_{1,j} + i\varepsilon_{2,j}, \quad 1 \leq j \leq n \quad (3)$$

where the s_m^* 's are unknown complex periodic signals with unknown fundamental frequencies f_m^* and the $\varepsilon_{1,j}$'s and $\varepsilon_{2,j}$'s have the same properties as previously. Some authors have already studied this problem with particular signals s_m^* . In the case where $M = 2$, Gassiat and Lévy-Leduc propose, in [7], semiparametric estimators of the two frequencies for a large class of signals, which are consistent and asymptotically Gaussian. When the ratio of the periods is not rational, the estimators are proved to be asymptotically independent and efficient.

Let us now give an outline of the paper. In Section II, we present our estimation method. We describe step by step in Section III a way to implement the method on a synthetic example. We apply in Section IV our algorithm to laser vibrometry signals and compare this approach with a classical one in such a field: the *microdoppler*. We show on several examples that our algorithm leads, as expected, to an accurate estimation of f^* at lower SNRs than the microdoppler. An extension of the algorithm to the estimation of several frequencies in model (3) is also proposed.

In the Appendix, we give a sketch of the proof of the asymptotic properties of our estimator, which is consistent and asymptotically efficient in an appropriate asymptotic framework.

II. ESTIMATION METHOD

A. Definition of the Estimator

We consider the model proposed in (2). We assume that the restriction of s^* to $[0, 1/f^*]$ is square integrable and smooth enough such that

$$(S1) \sum_{k \in \mathbb{Z}} k^2 |c_k(s^*)| < +\infty$$

where $(c_k(s^*), k \in \mathbb{Z})$ are the Fourier coefficients of s^* defined by

$$c_k(s^*) = f^* \int_0^{1/f^*} \exp(-2ik\pi f^* t) s^*(t) dt, \quad \forall k \in \mathbb{Z}.$$

If the unknown periodic function s^* is approximated by a trigonometric polynomial

$$s(t) = \sum_{k=-K}^K a_k e^{2ik\pi f t} \quad (4)$$

estimating s^* in a least-squares sense amounts to computing the coefficients $(a_k, -K \leq k \leq K)$ and the frequency f that minimizes $\|\mathbf{x} - \mathbf{s}\|^2$, where $\mathbf{x} = (x_j, 1 \leq j \leq n)$, $\mathbf{s} = (s(j\Delta), 1 \leq j \leq n)$ and $\|y\|^2 = \sum_{j=1}^n |y_j|^2$ for $y = (y_1, \dots, y_n) \in \mathbb{C}^n$.

For any given frequency $f \in \mathbb{R}_+$, let

$$d_k(f) = \frac{1}{n} \sum_{j=1}^n e^{-2ik\pi j f \Delta} x_j. \quad (5)$$

It is well-known (see [17]) that, for any K , $\sum_{j=1}^n |x_j - \sum_{k=-K}^K d_k(f) e^{2ik\pi j f \Delta}|^2$ is approximately equal, for large n , to

$$\min_{(a_k)} \left(\sum_{j=1}^n \left| x_j - \sum_{k=-K}^K a_k e^{2ik\pi j f \Delta} \right|^2 \right).$$

Then, for any $f \in \mathbb{R}_+$, the empirical variance of the observed sequence $n^{-1} \sum_{j=1}^n |x_j|^2$ can approximately be decomposed into

$$\frac{1}{n} \sum_{j=1}^n \left| x_j - \sum_{k=-K}^K d_k(f) e^{2ik\pi j f \Delta} \right|^2 + \sum_{k=-K}^K |d_k(f)|^2. \quad (6)$$

In other words, for every $K \geq 1$, the least-squares estimation of f^* is almost achieved by maximizing the so-called *cumulated periodogram*, which is defined for every $f \in \mathbb{R}_+$, by

$$\Lambda_{K,n}(f) = \sum_{k=-K}^K |d_k(f)|^2. \quad (7)$$

Of course, when dealing with sampled data, we estimate f^* in the interval $\mathcal{F} = (0, (2\Delta)^{-1}]$. Then, for every $K \geq 1$, f^* is estimated by

$$\hat{f}_n(K) = \text{Arg sup}_{f \in \mathcal{F}} \Lambda_{K,n}(f). \quad (8)$$

Without any information on the number of harmonics, K has also to be estimated. For any $f \in \mathcal{F}$, $\Lambda_{K,n}(f)$ clearly increases with K . Furthermore,

$$\Lambda_{2K,n}\left(\frac{f}{2}\right) = \Lambda_{K,n}(f) + \sum_{p=-K}^{K-1} |d_{2p+1}(f)|^2. \quad (9)$$

Thus, $\Lambda_{2K,n}(f^*/2) \geq \Lambda_{K,n}(f^*)$, and maximizing $\Lambda_{K,n}(f)$ with respect to both f and K will always lead to overestimate K . This overestimation of K may eventually lead to underestimate f^* (an example is given Section III-B). To avoid this overestimation of K , we will adopt a penalization approach where the estimated number of harmonics $\hat{K}_n$ and the estimated fundamental frequency $\hat{f}_n$ are defined by

$$\hat{K}_n = \underset{1 \leq K \leq K_{\text{Max}}}{\text{Arg min}} \left\{ -\Lambda_{K,n}(\hat{f}_n(K)) + \beta K \right\} \quad (10)$$

$$\hat{f}_n = \hat{f}_n(\hat{K}_n) \quad (11)$$

where β is a positive number to choose conveniently, and K_{Max} is an *a priori* upper bound for K .

We remark that

$$\hat{K}_n = \underset{1 \leq K \leq K_{\text{Max}}}{\text{Arg min}} \left\{ \frac{1}{n} \sum_{j=1}^n |x_j|^2 - \Lambda_{K,n}(\hat{f}_n(K)) + \beta K \right\}.$$

Then, by (6), $\hat{f}_n$ is approximately the penalized least-squares estimator of f^* .

B. Statistical Properties of the Estimator

Asymptotic properties of this estimator of f^* can be obtained in a very general asymptotic framework. As usual, the choice of this asymptotic framework is not motivated by practical consideration but by theoretical and technical reasons. This asymptotic framework is detailed in the Appendix. We briefly explain here our choice.

First, we assume that we have no *a priori* knowledge of bounds for the frequency f^* . Then, the estimator of f^* belongs to an interval $[F_{1,n}, F_{2,n}]$ which asymptotically covers $\mathbb{R}_+$. In order to avoid aliasing, we also have to consider that $\Delta = \Delta_n$ goes to 0 when n goes to infinity. We consider that the number of harmonics K used for approximating the unknown periodic function is not bounded. Then, $K \leq K_{\text{Max}} = K_n$, where K_n goes to infinity. For technical reasons, the penalization parameter $\beta = \beta_n$ decreases to 0 for large n .

In this asymptotic framework, we prove in the Appendix that $\hat{f}_n$ is consistent and asymptotically efficient. More precisely, we prove that

$$\lim_{\delta \rightarrow +\infty} \lim_{n \rightarrow +\infty} P\left(n^{3/2} \Delta_n |\hat{f}_n - f^*| > \delta\right) = 0$$

which means that $\hat{f}_n$ converges to f^* at the rate $1/(n^{3/2} \Delta_n)$. We also prove that

$$n^{3/2} \Delta_n (\hat{f}_n - f^*) \rightarrow \mathcal{N}(0, I^{-1}), \text{ in distribution}$$

where

$$I = \frac{1}{12 f^{*2} \sigma^2} \int_0^{1/f^*} |s^{*'}|^2(t) dt = \frac{\pi^2}{3\sigma^2} \sum_{k \in \mathbb{Z}} k^2 |c_k(s^*)|^2.$$

Remark: These results are similar to those obtained by Quinn and Thomson in [17], who prove in model (1), that

$$n^{3/2}(\hat{f}_n - f^*) \rightarrow \mathcal{N}(0, I^{-1}), \text{ in distribution.}$$

The term Δ_n does not appear in their result because they have chosen a sampling period equal to 1. Indeed, their asymptotic framework allows this assumption: *i)* The unknown periodic function has a finite number of harmonics, and *ii)* the frequency which they aim to estimate belongs to a bounded interval.

These asymptotic properties give some theoretical justification to our method but do not give any information on how to choose, in practice, the different parameters involved in the definition of $\hat{f}_n$. We propose in the next section a practical implementation of the method.

III. PRACTICAL IMPLEMENTATION ILLUSTRATED ON AN EXAMPLE

Obviously, the number of observations n is fixed in a practical situation. Then, we will use the notations Λ_K , $\hat{f}$, and $\hat{K}$ instead of $\Lambda_{K,n}$, $\hat{f}_n$, and $\hat{K}_n$.

The implementation of the previous method clearly raises up two main problems: the numerical maximization of Λ_K with respect to f and the choice of the penalization parameter β in (10). Nevertheless, we will explain step by step how to overcome these difficulties and illustrate our method on a simple synthetic example, motivated by some laser vibrometry application.

Let us first describe the laser vibrometry principle (see [14] for more details). A laser is assumed to emit a continuous wave that can be written $e(t) = \exp(2i\pi ct/\lambda)$, where c is the speed of light, and λ is the laser wavelength (namely $1.5 \mu\text{m}$). After emission, reflection on the illuminated vibrating object, reception, and demodulation, we receive a signal of the form (2). When the vibrating object consists of M punctual reflectors vibrating in a sinusoidal way, s^* can be written as follows:

$$s^*(t) = \sum_{m=1}^M a_m \exp\left(\frac{4i\pi\alpha_m}{\lambda} \cos(2\pi f^* t + \varphi)\right)$$

where a_m is the amplitude of the signal reflected by the reflector number m , α_m its vibration amplitude, f^* is the vibration frequency of the vibrating object, and $\varphi \in \mathbb{R}$. Thus, s^* is a periodic function with frequency f^* .

Note that this model is only used for simulating a periodic signal. Indeed, the proposed algorithm is adapted to any complex periodic signals (even to those having many harmonics) and never uses the parametric form of the laser vibrometry signals.

Note also that the following study focuses on a particular example which is just here to point out the pitfalls one can be faced with. There is no Monte Carlo experiment in this section. It is deferred to Section IV.

A. Presentation of the Synthetic Data

In this example, we aim to estimate the frequency of the following simulated complex valued signal:

$$s^*(t) = a \exp(ic \cos(2\pi f^* t)) \quad (12)$$

where $a = 0.06$, $c = 334$ and $f^* = 40$ Hz from the data: $x_j = s^*(j/n) + \varepsilon_{1,j} + i\varepsilon_{2,j}$, $1 \leq j \leq n$. Thus defined, s^* is

Fig. 1. Synthetic signal s^* . (a) Real part of $(s^*(j/n), 1 \leq j \leq 10000)$. (b) DFT spectrum of s^* . (c) Zoom of (b).

a periodic signal of frequency 40 Hz with about 334 (positive) harmonics at 40, 80, 120, $\dots$, 13360 Hz.

Fig. 1(a) displays the real part of $(s^*(j/n), j = 1, \dots, 10000)$ with $n = 2^{18}$ (we do not represent the complete sequence s^* since this signal is periodic). Fig. 1(b) displays the squared modulus of the discrete Fourier transform (DFT) of s^* defined by

$$I_{s^*}(q) = \frac{1}{n} \left| \sum_{j=1}^n e^{-2i\pi qj/n} s^* \left(\frac{j}{n} \right) \right|^2, \quad 0 \leq q \leq n-1.$$

We represent the sequence $I_{s^*}(q)$ only for $1 \leq q \leq 20000$ since this sequence is almost null for $q > 13360$. Furthermore, we see that aliasing is avoided with this value of n since $13360 \ll n/2$. Fig. 1(c) shows that most of the energy is contained in the even harmonics, that is, in the multiples of 80 Hz. Thus, the estimation of f^* will strongly depend on the number K . Indeed, if K is not large enough, the maximization of Λ_K yields 80 Hz instead of 40 Hz.

The observed sequence $(x_j, 1 \leq j \leq n)$ is obtained by adding a complex white noise to s^* :

$$x_j = s^* \left(\frac{j}{n} \right) + \varepsilon_{1,j} + i\varepsilon_{2,j}, \quad 1 \leq j \leq n \quad (13)$$

where $(\varepsilon_{1,j})$ and $(\varepsilon_{2,j})$ are independent Gaussian random variables with zero-mean and unit variance. For a signal s^* defined in (12), we define the SNR by $\text{SNR} = 10 \log_{10}(a^2/2)$. Thus, $\text{SNR} = -27$ dB in this example.

Fig. 2 displays the real part of the sequence $(x_j, 1 \leq j \leq 10000)$ and the squared modulus of the DFT of (x_j) (up to 20000). The SNR is very low in this example. Then, the original signal s^* and its spectrum cannot be visually detected.

B. Maximization of Λ_K With Respect to f

For any $K \geq 1$, it is impossible to maximize $\Lambda_K(f)$ with respect to f in a closed form. A numerical solution is to maximize Λ_K on a grid: For any $f \in \mathbb{R}$ and for any $K \geq 1$, we define

$$\tilde{\Lambda}_K(f) = \sum_{k=-K}^K |\hat{X}(\lfloor kf \rfloor)|^2$$

Fig. 2. Observed signal x . (a) Real part of $x = (x_j, 1 \leq j \leq 10000)$. (b) DFT spectrum of x . (c) Zoom of (b).

Fig. 3. Cumulated periodogram $\tilde{\Lambda}_K$ for different values of K . (a) $K = 100$. (b) $K = 500$. (c) $K = 700$.

where $\lfloor u \rfloor$ is the closest integer to the real number u , and $\hat{X}$ is the classical DFT of x defined by

$$\hat{X}(l) = \frac{1}{n} \sum_{j=1}^n e^{-2i\pi jl/n} x_j, \quad 0 \leq l \leq n-1.$$

Then, the use of $\tilde{\Lambda}_K$ instead of Λ_K will strongly reduce the computational cost of the algorithm since only the DFT of the observed signal x is computed by FFT.

Let $(f_m, 1 \leq m \leq M)$ be the elements of a regular grid with $f_{m+1} - f_m \leq 1/K$. This ensures that $\lfloor kf \rfloor = \lfloor kf_m \rfloor$, for any $f \in [f_m, f_{m+1})$ and any $1 \leq k \leq K$. Finally, for any $K \geq 1$, we can compute an estimate of f^* defined by

$$\hat{f}(K) = \underset{f \in \{f_m\}_{1 \leq m \leq M}}{\text{Arg sup}} \tilde{\Lambda}_K(f).$$

We plot in Fig. 3 the sequence $(\tilde{\Lambda}_K(f_m))$ obtained with $K = 100, K = 500$, and $K = 700$. Here, the f_m 's belong to the interval $[1 \text{ Hz}, 100 \text{ Hz}]$ with $f_{m+1} - f_m = 0.001 \text{ Hz}$.

TABLE I
ESTIMATED FREQUENCY AS A FUNCTION OF K

K	100;207	208;663	664;700
$\hat{f}(K)$	80	40	20

We summarize the estimations $\hat{f}(K)$ in Table I. Remember that the number of significant positive harmonics of s^* is 334 in this example. Then, we can see that f^* is well estimated when K is around 334 ($K \in [208; 663]$). A smallest value of K yields $\hat{f}(K) = 2f^*$, whereas a largest value yields $\hat{f}(K) = f^*/2$.

This “doubling/halving effect” can be explained as follows. First, overestimation of f^* when K is small with respect to the number of significant positive harmonics is due to the particular shape of the spectrum. Indeed, most of the energy is contained in the even harmonics, that is, in the multiples of 80 Hz. On the other hand, a large value of K can eventually lead to underestimate f^* . Indeed, without noise, we can remark that $\Lambda_{2K}(f^*/2) = \Lambda_{2K}(f^*)$. Thus, in the presence of white noise, $\Lambda_{2K}(f^*/2)$ and $\Lambda_{2K}(f^*)$ are two random variables with the same probability distribution, and $P(\hat{f}(K) = f^*/2) = P(\hat{f}(K) = f^*)$. Then, any of the value f^* or $f^*/2$ may be selected depending on the noise outcome. More generally, if $mK^* \leq K \leq mK^* + 1$ where K^* represents the number of significant positive harmonics

$$P\left(\hat{f}(K) = \frac{f^*}{m}\right) \approx P\left(\hat{f}(K) = \frac{f^*}{(m-1)}\right) \approx \dots \approx P(\hat{f}(K) = f^*).$$

The next section is devoted to a model selection approach for choosing the number of harmonics K .

C. Choice of β and the Computation of $\hat{K}$

For every $K \geq 1$, let

$$J(K) = \frac{1}{n} \sum_{j=1}^n |x_j|^2 - \tilde{\Lambda}_K(\hat{f}(K)). \quad (14)$$

According to (6), (7), and (14), $J(K)$ is an approximation of the residual variance when a trigonometric polynomial of degree K is fitted to the observed sequence $\mathbf{x}$. A first method for selecting $\hat{K}$ aims at locating the point where the slope of the residual variance changes.

On the other hand, a penalization method consists in defining

$$\hat{K}(\beta) = \underset{1 \leq K \leq K_{\text{Max}}}{\text{Arg min}} \{J(K) + \beta K\} \quad (15)$$

where the penalization parameter β yields a tradeoff between the fit with the data and the number of harmonics.

The following Lemma emphasizes the link between these two methods.

Lemma 1: There exist two sequences $K_1 = 1 < K_2 < \dots$, and $\beta_0 = +\infty > \beta_1 > \dots$, with

$$\beta_p = \max_{K_p < K \leq K_{\text{Max}}} \frac{J(K) - J(K_p)}{K_p - K} = \frac{J(K_{p+1}) - J(K_p)}{K_p - K_{p+1}}, \quad p \geq 1$$

and such that $\forall \beta \in (\beta_p; \beta_{p-1}]$, $\hat{K}(\beta) = K_p$.

Fig. 4. (a) Sequence $(K, J(K))$. (b) Sequence $(K_p, J(K_p))$.

It is straightforward that $\beta_0 = +\infty > \beta_1 > \dots$ and that for any β in $(\beta_p; \beta_{p-1}]$, K_p minimizes $J(K) + \beta K$. Then, we deduce from Lemma 1 that the length of the interval $(\beta_p; \beta_{p-1}]$ is the second derivative of the sequence $(K_p, J(K_p))$. In other words, the number K that maximizes the second derivative of the sequence $(K_p, J(K_p))$ is $K_{\hat{p}}$, where $\hat{p}$ maximizes $l_p = \beta_{p-1} - \beta_p$. Then, we estimate the unknown frequency f^* by $\hat{f} = \hat{f}(K_{\hat{p}})$.

Finally, the estimation algorithm can be summarized as follows:

- 1) For $K = 1, \dots, K_{\text{Max}}$, compute $J(K)$ using (14).
- 2) Compute the sequences (K_p) and (β_p) using the following recursion: We start with $K_1 = 1$ and $\beta_0 = +\infty$, and we set at step p

$$\beta_p = \max_{K_p < K \leq K_{\text{Max}}} \frac{J(K) - J(K_p)}{K_p - K} \quad \text{and} \quad K_{p+1} = \underset{K_p < K \leq K_{\text{Max}}}{\text{Arg max}} \frac{J(K) - J(K_p)}{K_p - K}. \quad (16)$$

- 3) Let $\hat{p} = \underset{p \geq 1}{\text{Arg max}} \{l_p\}$ where $l_p = \beta_{p-1} - \beta_p$. Then, set $\hat{K} = K_{\hat{p}}$ and $\hat{f} = \hat{f}(\hat{K})$.

We recommend to choose in practice a large value for K_{Max} with respect to the number of positive harmonics of the signal. Indeed, the penalty term avoids overestimating the number of significant harmonics of the signal. On the other hand, we have seen that f^* can be overestimated if K_{Max} is too small.

Fig. 4(a) displays the sequence $(K, J(K))$, and Fig. 4(b) displays the associated sequence $(K_p, J(K_p))$ obtained from the synthetic signal (x_j) . We clearly see on this figure that the residual variance decreases much more for $1 \leq K \leq 339$ than for $K > 339$. In other words, a good tradeoff between the fit with the data and the number of harmonics is obtained with $K = 339$, that is, for the value of K that maximizes the second derivative of the sequence $(K_p, J(K_p))$.

TABLE II
VALUES OF $\hat{K}(\beta)$, $\hat{f}(\hat{K}(\beta))$ AS FUNCTIONS OF β
AND THE VALUES OF $l_p = \beta_{p-1} - \beta_p$

$(\beta_p; \beta_{p-1}]$ ($\times 10^5$)	$\hat{K}(\beta)$	$\hat{f}(\hat{K}(\beta))$	l_p ($\times 10^6$)
$(2.98, +\infty)$	0	0	$+\infty$
$(2.85, 2.98]$	20	16.47	1.29
$(2.71, 2.85]$	22	32.94	1.44
$(2.67, 2.71]$	167	80	0.42
$(1.74, 2.67]$	339	40	9.2
$(1.55, 1.74]$	344	40	1.92
$(1.54, 1.55]$	678	20	0.06
$(1.38, 1.54]$	688	20	1.7
$(0.77, 1.38]$	699	20	6.05

Fig. 5. BIC and AIC criteria.

Table II displays the numbers (K_p), the estimated frequencies ($\hat{f}(K_p)$), and the lengths (l_p) of the intervals $(\beta_p; \beta_{p-1}]$. Here, the maximization of l_p yields $\hat{K} = 339$ and $\hat{f} = 40$ Hz, which is the true value of f^* .

Let us now compare on this example our method with some classical penalization methods such as the Bayesian Information Criterion (BIC) [19] or Akaike Information Criterion (AIC) [2], where

$$\text{BIC}(K) = 2 \log(J(K)) + \frac{\log(n)}{n} K$$

$$\text{AIC}(K) = 2 \log(J(K)) + \frac{2}{n} K.$$

We can see in Fig. 5 that the penalties given by the BIC criterion are too heavy, since $\hat{K}_{\text{BIC}} = 0$. On the other hand, the penalties on the AIC criterion are too weak, since $\hat{K}_{\text{AIC}} = 689$, and $\hat{f}_{\text{AIC}} = 20$ Hz.

IV. SOME MONTE CARLO EXPERIMENTS ON DIFFERENT SYNTHETIC VIBROMETRY SIGNALS

We will see in this section that the penalized cumulated periodogram provides an accurate estimation of f^* at SNR much smaller than those required by the microdoppler approach, which is a classical method used in laser vibrometry for specific signals like (12) (more details on the microdoppler method can be found in [12] and [18]). This is an important

TABLE III
COMPARISON OF THE RMSE FOR THE MICRODOPPLER (e_m)
AND THE PENALIZED CUMULATED PERIODOGRAM (e_p)
COMPUTED FROM FOUR DIFFERENT TEST SIGNALS

	SNR	-29	-24	-19	-14	-9
ex. 1	e_m	31.16	29.62	30.352	26.2	0
	e_p	8.25	0	0	0	0
ex. 2	e_m	29.97	27.19	29.15	25.69	0.497
	e_p	7.61	0.0004	0.0004	0.0004	0.0004
ex. 3	e_m	31.00	29.54	30.20	24.21	0
	e_p	5.66	0.002	0.002	0.002	0.002
ex. 4	e_m	42.3	36.33	36.05	18.97	6.32
	e_p	13.611	0.0007	0.0006	0.0006	0.0006

point in practice, because it means that the vibration analysis can be achieved even if the vibrating object is far from the laser. Indeed, we remark that the further away the vibrating object is, the lower the SNR will be. For instance, in a real-world situation, an SNR of -30 dB corresponds to a distance of 50 km between the laser and the target.

A. Estimation of One Frequency

We propose several numerical comparisons of the two approaches obtained with different signals s^* and different values of SNR (see Table III). For each signal s^* and each value of SNR, we have simulated $L = 100$ observed series (x_j) satisfying (13) and estimated f^* by the two methods proposed previously. The quality of an estimator is quantified by computing the root mean squared error (RMSE) defined by $\sqrt{L^{-1} \sum_{i=1}^L (\hat{f}_i - f^*)^2}$, where $(\hat{f}_i)$ are the L estimations obtained for f^* . e_m and e_p will be the RMSE for the microdoppler and the penalized cumulated periodogram.

- 1) In this example, we set $s^*(t) = a \exp(ic \cos(2\pi f^* t))$ where $f^* = 40$ Hz, $c = 4\pi\alpha/\lambda$ with $\alpha = 40 \times 10^{-6}$, $\lambda = 1.5 \times 10^{-6}$, and $n = 2^{18}$. We can remark that in this case, the frequency f^* is an integer. Then, for SNR large enough, the RMSEs e_m and e_p can be equal to 0 since the true frequency f^* belongs to the grid (f_m) , where the estimator takes its values.
- 2) The parameters remain unchanged in this second example, but the unknown frequency $f^* = 46.5136$ Hz no longer belongs to the grid (f_m) , since $f_m - f_{m-1} = 0.001$. Thus, the RMSEs e_p and e_m cannot be 0. Here, the minimum value of the RMSE is $46.514 - 46.5136 = 0.0004$.
- 3) In this third example, the signal s^* is a sum of 12 elementary signals with different vibration amplitudes c_s : $s^*(t) = a \sum_{s=1}^{12} \exp(ic_s \cos(2\pi f^* t))$ where $f^* = 40$ Hz, $c_s = 4\pi\alpha_s/\lambda$ with $\alpha_s = 7s \times 10^{-6}$ for $1 \leq s \leq 12$, $\lambda = 1.5 \times 10^{-6}$, and $n = 2^{18}$. Here, the SNR is defined by $10 \log_{10}(12a^2/2)$. The simulations show that there is no degradation of the estimation in the sense that we obtain an accurate result for the estimation of f^* when $\text{SNR} \geq -24$ dB for the penalized cumulated periodogram and when $\text{SNR} \geq -9$ dB for the microdoppler.
- 4) In this example, we use $s_k(t) = a \sum_{s=1}^{12} \exp(ic_s \cos(2\pi f_k t))$ with the same parameters (α_s) , λ , and n , and for different fundamental frequencies (f_k) . The 100 frequencies (f_k) are randomly chosen with a

uniform distribution between 10 and 100 Hz. We conclude as usual that the penalized cumulated periodogram performs better than the microdoppler at small SNR.

Comments:

- 1) In the estimation of f^* , one can distinguish two types of errors: the large errors that correspond to multiples or submultiples of the frequency and finer estimation errors (such as those due to grid searching). Here, we propose to use a tool for evaluating the estimation errors (RMSE), which aggregate both type of errors for a sake of concision. However, despite this aggregation, we can say from the results that when the RMSE is equal to the minimum distance between f^* and the grid (see 0.0004 in example 2 for instance), then the error is only of the second type, which is not the case for larger value of RMSE for which the error is mainly of the first type.
- 2) Our values of RMSE cannot be compared with the "Cramer-Rao bound" (i.e., the inverse of the efficient Fisher information in a semiparametric framework), which is approximately equal to 10^{-9} in the different examples. Indeed, the asymptotic variance is only valid in a neighborhood of f^* , that is, when f^* is well estimated, without any "doubling/halving effect." Furthermore, this comparison is not relevant since the mesh of the grid, where $\hat{f}$ is computed, is much larger than $\sqrt{10^{-9}}$ Hz. Such a precision would have a very large computational cost and is not required in practice.

B. Estimation of Two Frequencies

The use of the penalized cumulated periodogram can be extended to model (3) for estimating M fundamental frequencies. We consider in this section the case $M = 2$. Here, s_1^* and s_2^* are periodic functions with frequencies f_1^* and f_2^* .

A general condition of identifiability is given in [7]: Without any assumption on the signals s_1^* and s_2^* , the frequencies f_1^* and f_2^* are not identifiable when their ratio is rational. Nevertheless, if s_1^* and s_2^* are both of the form given in (12), it can be proved that the model (3) is identifiable if this ratio is not an integer.

For any frequency f , let

$$I_n(f) = \frac{1}{n} \left| \sum_{j=1}^n e^{(-2i\pi j f)/n} x_j \right|^2$$

and then, for any K_1, K_2 , an estimator of (f_1^*, f_2^*) is

$$(\hat{f}_1(K_1), \hat{f}_2(K_2)) = \underset{(f_1, f_2)}{\text{Arg sup}} \sum_{f \in \mathcal{F}(f_1, f_2)} I_n(f)$$

$$\text{where } \mathcal{F}(f_1, f_2) = \bigcup_{m=1}^2 \{kf_m, -K_m \leq k \leq K_m\}.$$

The implementation of this method mainly raises two difficulties: the numerical maximization with respect to (f_1, f_2) and the choice of K_1 and K_2 . We propose the following procedure:

- 1) Estimate f_1^* by maximizing $\Lambda_{K_1}(f_1)$, where the number of harmonics K_1 is chosen by using the penalization procedure described above.

TABLE IV
ROOT MEAN SQUARED ERRORS FOR THE MICRODOPPLER (e_{m1}, e_{m2})
AND THE PENALIZED CUMULATED PERIODOGRAM (e_{p1}, e_{p2})

	SNR	-29	-24	-19	-14	-9
ex.1	e_{m1}	27.17	19.23	5.79	0	0
	e_{m2}	34.89	29.64	11.51	0	0
	e_{p1}	10.24	5.30	0	0	0
ex.2	e_{p2}	15.13	6.59	0	0	0
	e_{m1}	24.29	16.82	7.72	0	0
	e_{m2}	33.59	30.46	37.14	47	47
ex.3	e_{p1}	8.78	3.66	0	0	0
	e_{p2}	23.91	17.55	1.6	0	0
	e_{m1}	24.73	26.06	8.48	1.73	0
	e_{m2}	43.88	41.18	24.61	9	0
	e_{p1}	5.38	2.85	4.93	0	0
	e_{p2}	33.71	36.37	9.99	0	0

- 2) Estimate f_2^* by maximizing $\Psi_{K_2}(f_2)$, where

$$\Psi_{K_2}(f_2) = \sum_{g \in \mathcal{F}(\hat{f}_1, f_2)} I_n(g) - \Lambda_{\hat{K}_1}(\hat{f}_1).$$

The number of harmonics K_2 is chosen by using the penalization procedure described above.

To extend the microdoppler technique is straightforward for estimating the two frequencies f_1^* and f_2^* and is not detailed here.

The results of some numerical experiments are summarized in Table IV. Like previously, we have computed the RMSE with both algorithms. We have simulated 100 series x of length $n = 2^{18}$ satisfying

$$x_j = s_1^* \left(\frac{j}{n} \right) + s_2^* \left(\frac{j}{n} \right) + \varepsilon_{1,j} + i\varepsilon_{2,j}, \quad 1 \leq j \leq n$$

for each example where $\varepsilon_{1,j}$ and $\varepsilon_{2,j}$ are standard Gaussian random variables.

In the first two examples, the signal s^* is the sum of two signals s_1^* and s_2^* , where $s_1^*(t) = a_1 \exp(ic_1 \cos(2\pi f_1^* t))$ and $s_2^*(t) = a_2 \exp(ic_2 \cos(2\pi f_2^* t))$, and where $f_1^* = 30$ Hz, $f_2^* = 43$ Hz, $c_1 = 4\pi\alpha_1/\lambda$, $c_2 = 4\pi\alpha_2/\lambda$ with $\alpha_1 = 40 \times 10^{-6}$, $\alpha_2 = 45 \times 10^{-6}$ and $\lambda = 1.5 \times 10^{-6}$. Here, the SNR is defined by $\text{SNR} = 10 \log_{10} ((a_1^2 + a_2^2)/2)$.

In the first example, $a_1 = a_2$. We can remark that both algorithms are efficient for estimating the two frequencies, but the cumulated periodogram requires a lower SNR than the microdoppler.

In the second example, $a_1 = \sqrt{10}a_2$. In this case, the microdoppler is unable to estimate the frequency of the signal of lower power: For $\text{SNR} \geq -14$ dB, the estimation of f_2^* is 90 Hz instead of 43 Hz.

In the third example, $s_1^*(t) = a \sum_{k=1}^{20} \exp(ic_k \cos(2\pi f_1^* t))$ and $s_2^*(t) = a \sum_{k=1}^{20} \exp(id_k \cos(2\pi f_2^* t))$, where $f_1^* = 20$ Hz, $f_2^* = 30$ Hz, $c_k = 4\pi\alpha_k/\lambda$, and $d_k = 4\pi\beta_k/\lambda$, with $\alpha_k = (10+4k) \times 10^{-6}$, $\beta_k = (20+3k) \times 10^{-6}$, and $\lambda = 1.5 \times 10^{-6}$. Here, the SNR is defined by $\text{SNR} = 10 \log_{10} (40a^2/2)$.

V. CONCLUSION

We have proposed in this paper a method for estimating the frequency of unknown periodic functions corrupted by Gaussian white noise based on the penalized cumulated periodogram. By

using a semiparametric approach, we have proved that our frequency estimator has the best asymptotic properties that one can expect, and we have implemented it. We have applied our algorithm to laser vibrometry signals and have shown that our method leads to an accurate frequency estimation at a lower SNR than usual methods. We have also proposed an extension of our method to the estimation of the frequencies of two periodic functions. We think that it could be extended to the estimation of more than two frequencies.

APPENDIX

We consider the asymptotic framework such that Δ_n , $F_{1,n}$, and β_n go to 0, whereas $F_{2,n}$ and K_n go to infinity with n . Asymptotic results require precisely how these sequences vary with n . Hence, we will adopt the following asymptotic framework to prove the consistency and efficiency of $\hat{f}_n$.

A1)

$$\lim_{n \rightarrow +\infty} \Delta_n = 0, \quad \lim_{n \rightarrow +\infty} n\Delta_n^2 = +\infty.$$

A2)

$$\lim_{n \rightarrow +\infty} n\Delta_n F_{1,n} = +\infty, \quad \lim_{n \rightarrow +\infty} F_{2,n} = +\infty, \quad \lim_{n \rightarrow +\infty} \Delta_n F_{2,n} = 0.$$

A3)

$$\begin{aligned} \lim_{n \rightarrow +\infty} K_n \Delta_n F_{2,n} &= 0, \quad \lim_{n \rightarrow +\infty} \frac{K_n}{F_{2,n}} = 0, \quad \lim_{n \rightarrow +\infty} K_n F_{1,n} = 0 \\ \lim_{n \rightarrow +\infty} \frac{K_n^{5/2}}{n F_{1,n} \Delta_n} &= 0, \quad \lim_{n \rightarrow +\infty} \frac{K_n^2}{n \Delta_n^2} = 0, \quad \lim_{n \rightarrow +\infty} \frac{n \Delta_n}{K_n^5} = 0. \end{aligned}$$

A4)

$$\begin{aligned} \lim_{n \rightarrow +\infty} \beta_n &= 0, \quad \lim_{n \rightarrow +\infty} \frac{\beta_n n \Delta_n F_{1,n}}{K_n} = +\infty \\ \text{there exists } \delta > 0 \text{ s.t. } n\beta_n^2 &= O(n^\delta). \end{aligned}$$

This leads to the following asymptotic result.

Theorem 1: Assume **(S1)**; then, under **A1)–A4)**, the estimated frequency $\hat{f}_n$ defined by (10) and (11) satisfies

$$\lim_{\delta \rightarrow +\infty} \lim_{n \rightarrow +\infty} P\left(n^{3/2} \Delta_n |\hat{f}_n - f^*| > \delta\right) = 0$$

and the following asymptotic linearization property:

$$n^{3/2} \Delta_n (\hat{f}_n - f^*) = -\frac{I^{-1}}{\sigma^2 n^{3/2}} \sum_{j=1}^n \left(\frac{n}{2} - j\right) \frac{s^{*'}(j\Delta_n)}{f^*} \varepsilon_j + r_n$$

where

$$I = \frac{1}{12 f^{*2} \sigma^2} \int_0^{1/f^*} |s^{*'}|^2(t) dt$$

and where the remainder term r_n converges to 0 in probability.

We give a sketch of the proof of the consistency and asymptotic efficiency of the estimator $\hat{f}_n$ (for more details, see [12]). For a sake of simplicity, we just consider real-valued observations, which means that the observations (x_j) that we consider are the real part of the (x_j) satisfying model (2).

The first step consists of proving the consistency of $\hat{f}_n$ defined in (11). For this, we need to use the following Lemma, which comes from the fact that s^* is periodic of frequency f^* and not $2f^*$ or any multiple of f^* .

Lemma 2: There exists $K^* \geq 1$, h in $]0,1[$ such that for any $m \geq 2$

$$\sum_{k=1}^K |c_{mk}(s^*)|^2 < (1-h) \sum_{k=1}^K |c_k(s^*)|^2, \quad \text{for any } K \geq K^*$$

and

$$\sum_{k=1}^{K^*-1} |c_{mk}(s^*)|^2 \geq (1-h) \sum_{k=1}^{K^*-1} |c_k(s^*)|^2.$$

Remark: The previous Lemma implies that $c_{K^*}(s^*) \neq 0$. This statement would naturally be satisfied if s^* was a trigonometric polynomial of degree K^* .

We also need to prove that the random part of $\Lambda_{K,n}$ tends to zero in probability with a precise rate.

For every $f \in \mathbb{R}_+$ and for every $K \geq 1$, let

$$G_{K,n}(f) = \frac{1}{n^2} \sum_{k=1}^K \left| \sum_{j=1}^n e^{2i\pi k j f \Delta_n} s^*(j\Delta_n) \right|^2$$

$$\text{and } \eta_{K,n}(f) = \Lambda_{K,n}(f) - G_{K,n}(f)$$

where $\Lambda_{K,n}$ is defined in the Appendix by

$$\Lambda_{K,n}(f) = \frac{1}{n^2} \sum_{k=1}^K \left| \sum_{j=1}^n e^{2i\pi k j f \Delta_n} x_j \right|^2.$$

Indeed, since the (x_j) are real, $\Lambda_{K,n}$ defined in (7) can be rewritten as follows:

$$\Lambda_{K,n}(f) = n^{-1} \left| \sum_{j=1}^n x_j \right|^2 + 2n^{-2} \sum_{k=1}^K \left| \sum_{j=1}^n e^{2i\pi k j f \Delta_n} x_j \right|^2.$$

Therefore, taking the definition of $\Lambda_{K,n}$ given in (7) or the one that we take in the Appendix does not change the definition of $\hat{K}_n$ and, hence, of $\hat{f}_n$.

We can consider $G_{K,n}$ as the "deterministic component" of $\Lambda_{K,n}$ and $\eta_{K,n}$ as the random fluctuations of $\Lambda_{K,n}$. The following Lemma shows that the sequence $(\eta_{K,n})$ is asymptotically negligible in probability.

Lemma 3: Under **(S1)** and **A1)–A3)**, for all $x_n > 0$ and for all $K \in \{1, \dots, K_n\}$

$$P\left(\sup_{f \in \mathcal{F}_n} |\eta_{K,n}(f)| \geq x_n\right) \leq C' x_n n \sqrt{n} \log(n) \exp(-C n x_n^2)$$

for some positive constants C and C' , and $\mathcal{F}_n = [F_{1,n}, F_{2,n}]$.

This result is a straightforward extension of Lemma 1 in [7].

With these statements, we prove that $P(\hat{K}_n < K^*)$ goes to zero when n goes to infinity, and we can conclude that $\hat{f}_n$ is consistent at some rate α_n . Then, this rate can be improved by using an argument similar to the one used in [17] and becomes $\rho_n/(n\Delta_n)$, where ρ_n tends to 0 when n becomes large.

The last step consists of obtaining an asymptotic linearization of $\hat{f}_n$. It comes from a Taylor expansion of $\Lambda_{K,n}$ around f^* . This linearization leads to the asymptotic normality of the estimator with the expected asymptotic variance I^{-1} , where I is defined in Section II-B.

ACKNOWLEDGMENT

The authors would like to thank the anonymous referees for their relevant remarks, the company Thalès Optronique, which has supported this study, and, more particularly, M. Prenat who proposed the subject.

REFERENCES

- [1] H. Akaike, "A new look at the statistical model identification," *IEEE Trans. Autom. Control*, vol. AC-19, no. 6, pp. 716–723, Dec. 1974.
- [2] —, "A Bayesian analysis of the minimum aic procedure," *Ann. Inst. Statist. Math.*, pt. A, vol. 30, pp. 9–14, 1978.
- [3] L. Birgé and P. Massart, "Gaussian model selection," *J. Eur. Math. Soc.*, vol. 3, pp. 203–268, 2001.
- [4] —, "A Generalized C_p Criterion for Gaussian Model Selection. Tech. Rep.," Publication de l'Université Paris VI, Paris, France, 2001.
- [5] R. Carmona, W.-L. Hwang, and B. Torrésani, *Practical Time-Frequency Analysis*, C. K. Chui, Ed. New York: Academic, 1998, Wavelet analysis and its application.
- [6] A. De Cheveigné and H. Kawahra. (1998) Multiple Period Estimation and Pitch Perception Model. [Online]. Available: <http://cite-seer.nj.nec.com/284277.html>
- [7] E. Gassiat and C. Lévy-Leduc, "Efficient Semiparametric Estimation of the Periods in a Superposition of Periodic Functions with Unknown Shape, Techn. Rep.," Prépublication de l'Université Paris XI, Orsay, France, 2003, submitted for publication.
- [8] G. K. Golubev, "Estimation of the period of a signal with an unknown form against a white noise background," *Problemy Peredachi Informatsii*, vol. 24, no. 4, pp. 38–52, 1988.
- [9] S. M. Kay, "A fast and accurate single frequency estimator," *IEEE Trans. Acoust., Speech, Signal Process.*, vol. 37, no. 12, pp. 1987–1989, Dec. 1989.
- [10] P. J. Kootsookos. (1999) A Review of the Frequency Estimation and Tracking Problems. Frequency Estimation and Tracking Project. CRC for Robust and Adaptive Systems, DSTO, Salisbury Site. [Online]. Available: www.itee.uq.edu.au/kootsoop/comparaison-t.pdf
- [11] M. Lavielle and E. Moulines, "Least-squares estimation of an unknown number of shifts in a time series," *J. Time Series Anal.*, vol. 21, no. 1, pp. 33–59, 2000.
- [12] C. Lévy-Leduc, *Estimation Semi-Paramétrique de la Fréquence de Fonctions Périodiques Dans Divers Modèles Statistiques : Théorie et Pratique*. Orsay, France: Université Paris-Sud, 2004.
- [13] B. C. Lovell and R. C. Williamson, "The statistical performance of some instantaneous frequency estimators," *IEEE Trans. Signal Process.*, vol. 40, no. 7, pp. 1708–1723, Jul. 1992.
- [14] M. Prenat, "Vibration modes and laser vibrometry performance in noise," in *Proc. Phys. Signal Image Process. Conf.*, Marseille, France, 2001.
- [15] B. G. Quinn and J. M. Fernandes, "A fast efficient technique for the estimation of frequency," *Biometrika*, vol. 78, no. 3, pp. 489–497, 1991.
- [16] B. G. Quinn and E. J. Hannan, *The Estimation and Tracking of Frequency*. Cambridge, U.K.: Cambridge Univ. Press, 2001, Cambridge Series in Statistical and Probabilistic Mathematics.
- [17] B. G. Quinn and P. J. Thomson, "Estimating the frequency of a periodic function," *Biometrika*, vol. 78, no. 1, pp. 65–74, 1991.
- [18] I. Renhorn, C. Karlsson, D. Letalick, M. Millnert, and R. Rutgers, "Coherent laser radar for vibrometry: Robust design and adaptive signal processing," *Proc. SPIE*, vol. 2472, pp. 23–30, 1995.
- [19] G. Schwarz, "Estimating the dimension of a model," *Ann. Statist.*, vol. 6, no. 2, pp. 461–464, 1978.
- [20] A. W. van der Vaart, *Asymptotic Statistics*. Cambridge, U.K.: Cambridge Univ. Press, 1998, Cambridge Series in Statistical and Probabilistic Mathematics.

Marc Lavielle was born in Paris, France, in 1961. He received the Ph.D. degree in mathematics from the University of Paris-Sud, Orsay, France, in 1990.

From 1987 to 1991, he was an Assistant Professor with the Universidad Central de Venezuela, Caracas, Venezuela. He is currently Professor at the University René Descartes, Paris. His research interests include statistical methods for signal processing, stochastic algorithms, geophysics, and biostatistics.

Céline Lévy-Leduc was born in Montpellier, France, in 1977. She received the Agrégation de Mathématiques degree in 2000 and is currently pursuing the Ph.D. degree with E. Gassiat and M. Lavielle at the University of Paris-Sud, Orsay, France.

She is an engineer with Thalès Optronique, Guyancourt, France. Her research interests include statistical methods for signal processing and semi-parametric estimation.